# Chapter 3

# Calculus; Motion at Constant Acceleration

## 3.1 Introduction

At the start of the book, I supplied the fundamental equation

PHYSICS - CALCULUS = NONSENSE

Calculus is not a new tool. It was developed more or less independently by Isaac Newton and Gottfried Wilhelm Leibniz nearly four centuries ago. Newton and Leibniz had a vigorous and somewhat pointless dispute as to who had developed what first. Recently, it became apparent that at about the same time an Austrian monk had also developed the Fundamental Theorem of Calculus. He tragically died, saving a young boy from drowning in a mountain stream, before he could publish his results. It is a curious fact that one of the allowed occupations of Samurai warriors in the Tokugawa period of Japan was abstract mathematics, a skill believed to be as useful as calligraphy or flower arranging. Some historians have made a case that this mathematical school independently developed, using a very different representation of mathematics, the basic ideas of calculus. What you should now have studied is Newtonian calculus, using the more sophisticated notation of Leibniz. As has been said before, Newton was a truly brilliant man, so he didn't worry whether or not his notation was easy to use or prone to introducing errors. Leibniz viewed himself as writing for mere mortals, and therefore thought carefully about how to make his notation easy to understand and unambiguous in employment.

In this chapter, I provide a short refresher on calculus and show a single application, sometimes described as *kinematics* or as *motion at constant acceleration*. As emphasized in the Introduction, this course assumes that you've already had enough calculus to be familiar with the integrals and derivatives of standard functions.

## 3.2 The Standard Functions

I first mention a few things you should already have heard about. The standard functions of interest are:

- The polynomial $f(x) = a + bx + cx^2 + \ldots$ .
- The trigonometric functions $\sin(ax)$ and $\cos(ax)$.
- The exponential $e^{ax}$, also written $\exp(ax)$.
- The natural logarithm $\log(ax)$, sometimes also written $\ln(ax)$ or $\log_e(ax)$.

For each of these functions, you should be able to take the integral and the derivative. Most of you will also have seen and derived formulas for integrals and derivatives of all sorts of other trig functions, hyperbolic trig functions, and strange functions of polynomials. More or less all of those other formulas are far less useful for this course, so they are not reviewed here.

There is one minor point with which some of you are unfamiliar, namely the use of $\exp(ax)$ as an alternative way to write the exponential $e^{ax}$. There is a sound reason for introducing $\exp(ax)$ for the exponential. The argument of an exponential can become complicated, in which case writing the argument as a superscript becomes ugly.

There is also a major point which you have surely all seen but whose significance was not always made apparent. That's the constant of integration. For example, if I integrate $ax$ with respect to x, taking the *indefinite integral* in which the limits of integration are not specified, I obtain

$$\int ax\ dx = \frac{ax^2}{2} + x_0 \tag{3.1}$$

and not

$$\int ax\ dx = \frac{ax^2}{2}. \tag{3.2}$$

In this equation, $x_0$ is a constant, the *constant of integration.* Your calculus preparation may not have stressed why constants of integration are important. Later in the chapter, I will show why constants of integration can be critically important, what they do, and how to determine their values. As another minor point, some of you were shown integrals being written as $\int dx\ ax$ while others will have seen $\int ax\ dx$. These are two ways of writing the same thing. Both forms are correct.

A few of you will have seen the misfortunate misrepresentation

$$\int ax. \tag{3.3}$$

This abomination is nonsense. The reason it is nonsense is that the symbol for integration is $\int d?$, not $\int$. In the correct symbol, the ? stands for the variable being integrated, which must be specified. If the variable is not specified, you have absolutely no idea with respect to what you are taking the integral. It might be $x$. It might be $a$. It might be $t$.

The derivative with respect to $t$ of a function $f(t)$ is often defined

$$\frac{df}{dt} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{(t+h) - t}. \tag{3.4}$$

The Fundamental Theorem of Calculus tells us that the integral is the antiderivative. In particular, for the definite integral of $df(t)/dt$ from $a$ to $b$, we have

$$\int_a^b \frac{df}{dt}\ dt = f(b) - f(a). \tag{3.5}$$

This form is termed the *definite integral* because the bounds of integrations are specified; the bounds are $a$ and $b$. What happened to the constant of integration? The subtraction on the right hand side of this equation cancels out the constant of integration in the indefinite integral.

If we take an integral of the function $g(x)$, we may think of $g(x)$ as giving us a smooth curve, which can be plotted as a function of $x$ by making $g(x)$ the distance of the function from the $x$-axis. The integral $\int g(x)\ dx$ is then the area under the curve. Area is a signed number. It maybe positive or negative. Thus, for example,

$$\int_0^\pi \cos(x)\ dx = 0, \tag{3.6}$$

because

$$\int_0^{\pi/2} \cos(x)\ dx = -\int_{\pi/2}^{\pi} \cos(x)\ dx. \tag{3.7}$$

The direction of integration is also a signed quantity, so that

$$\int_a^b \frac{df}{dt}\, dt = -\int_b^a \frac{df}{dt}\, dt. \tag{3.8}$$

In this equation, if you take the integral with respect to $t$ backwards rather than forwards along the $t$-axis, you get a number with the same magnitude, but the sign is opposite to the sign you get if you take the integral with respect to $t$ forwards along the $t$-axis.

One point not always emphasized in calculus courses is the dimension – dimension in the sense mass-length-time – of an integral or a derivative. The dimensions that come out of an integral or derivative actually follow immediately from equations 3.4 and 3.5. If we allow that $x$ and $t$ represent a spatial distance and time, then $\frac{d}{dx}$ has dimensions 1/length, while $\frac{d}{dt}$ has dimensions 1/time. For the same reasons $\int dx$ has dimensions length and $\int dt$ has dimensions time.

As an example, consider $dx/dt$, which gives the speed of an object whose location at time $t$ is $x(t)$. Speed has dimensions length/time. In $dx/dt$, $x$ has units length, so the 1/time must come from the $d/dt$. On the other hand, consider $\int \frac{dx}{dt}\, dt$, which has units length. The $\frac{dx}{dt}$ has units length/time, so to make things work $\int\ dt$ must have units time.

If $f(x)$ and $g(x)$ are both functions of the variable $x$, then you should recall how to take the derivatives of $f(x) + g(x)$, $f(x)g(x)$, and $(f(x))^n$. You also should know the chain rule, so that if $f$ is a function of $g$ and $g$ is a function of $x$, there is a process for taking $\frac{df(g(x))}{dx}$.

Finally, there is a standard rule for finding the maximum or minimum of a function $f(x)$, namely you look for the points where $\frac{df(x)}{dx} = 0$. This rule has several failures. In particular, $\frac{df(x)}{dx} = 0$ also locates all of the saddle points of a function. What is a saddle point? Imagine climbing up a hill, and part way up there is a short level stretch of ground before the hill starts climbing again. The short level stretch is a saddle point. The rule also fails at two sorts of points. You can understand these points by imagining the roof of the house. First, if the roof is entirely straight, but tilted, the slope of the roof is not zero anywhere including its two ends, but the low end is the minimum and the high end is the maximum. The derivative rule does not work at endpoints. In addition, if we have a traditional house roof with the peak in the middle, the maximum height of the roof is at the peak, but the derivative is not zero there. In fact, the derivative is not even defined at the peak of the roof, because the slope has a left-hand limit and a right-hand limit, and these two limits are not equal to each other.

The fact that a function is continuous does not mean that it is differentiable. You can have a continuous function that does not have a derivative at a point. Indeed, you can have a function that is continuous everywhere but because of its particular definition does not have a derivative at any point. Almost all physics functions are continuous and differentiable almost but sometimes not quite everywhere.

## 3.3 Motion at Constant Acceleration

We now turn to a simple use of calculus, namely *motion at constant acceleration.*

For the moment, we will consider motion in one dimension. Real space is three-dimensional; we'll come back to that. We have an object whose position along the $x$ axis is $x(t)$. The object is moving, so its position depends on time, that is, $x$ is indeed $x(t)$. The object then has a position $x(t)$, a velocity $v_x(t)$, and an acceleration $a_x(t)$. $x$ is the position of the object at some time, not the distance through which the object has moved. Note the subscript $x$. $v_x(t)$ and $a_x(t)$ refer to motions parallel to the $x$ axis. The velocity and acceleration are related to the position $x$ by

$$v_x(t) = \frac{dx(t)}{dt}, \tag{3.9}$$

$$a_x(t) = \frac{d^2x(t)}{dt^2}, \tag{3.10}$$

while the acceleration and the velocity are related to each other through a derivative as

$$a_x(t) = \frac{dv_x}{dt}. \tag{3.11}$$

By *motion at constant acceleration* I mean that the acceleration has a fixed value $A$ so that we may write for the object an *equation of motion*

$$\frac{d^2x}{dt^2} = A. \tag{3.12}$$

The equation of motion is the equation for the particle's acceleration, with a value inserted for the acceleration. In writing this equation, the value of the acceleration has been given a symbol $A$ that is not the same as the general symbol $a_x(t)$, so that we can tell them apart.

A perhaps-familiar example of motion at constant acceleration is falling motion in the absence of air resistance, as encountered on the surface of the Moon. For a fall at constant acceleration, we conventionally write

$$\frac{d^2z}{dt^2} = -g. \tag{3.13}$$

$z$ is the vertical axis, with $z$ increasing as we go up. Here $g$ describes the acceleration due to gravity. On Earth, to one significant figure, $g = 10$ m/s$^2$. However, if we were using a different set of units, such as cgs units or English units, $g$ would have a different numerical value, but the equation would still be correct. In one respect this equation is not the way things are normally done. $g$ is an algebraic symbol. Algebraic symbols may be positive or negative, but one usually writes them as though they were positive, and allows negative numerical values to appear at some stage in the solution process. Here we have taken $g$ to be a positive number, the minus sign (needed because objects fall in the downward direction) being inserted explicitly into the equation. That's not literally wrong, but it not the way things are usually done. It's better to let the signs fall out of the solutions, rather than wasting time guessing which sign each variable has, in order to arrange things so that algebraic symbols all stand for positive numbers.

Suppose we want to move from the acceleration to the velocity to the position of an object. We can do this by taking equation 3.12 and integrating it with respect to time. We'll treat the integrals of the two sides of the equation separately, and equate them at the end. The integral with respect to time of the left-hand-side of equation 3.12 is

$$\int_0^T dt\, \frac{d^2x}{dt^2} = v_x(T) - v_x(0). \tag{3.14}$$

One symbol, $t$, is used for the variable of integration, and a different symbol, $T$, is used for the time out to which the integral is taken. Nothing in this integral requires $T > 0$. A $T < 0$ is perfectly valid and allows integration backward in time, thus allowing us to calculate where the object came from as well as where it is going. By convention, $v_x(0)$ is written $v_{x0}$.

Why would you want to integrate backwards in time? One of the most powerful historical dating methods for ancient times uses the date and time of recorded eclipses. A solar eclipse lasts a few minutes and covers only a narrow geographical area. Lunar eclipses are seen everywhere on earth, and appear to last much longer than solar eclipses. One finds a record of an eclipse on a certain date and place in local time, then back calculates from today when an eclipse would have been seen in a particular place at about the right date. Sometimes identifying which eclipse the ancients were describing can be challenging. Once this is done, the exact date for an ancient civilization's calendar has now been determined relative to the dates in our calendar.

We can equally well integrate the right-hand-side of equation 3.12 with respect to time, obtaining

$$\int_0^T A\ dt = AT - 0 \tag{3.15}$$

What is the integral of equation 3.12 with respect to time? We have now done integrals of the two sides of the equation separately. Combining the above results, the integral is

$$v_x(T) - v_{x0} = AT \tag{3.16}$$

or the more familiar

$$v_x(T) = v_{x0} + AT. \tag{3.17}$$

We now have the velocity as a function of $v_{x0}$, $A$, and time $T$. Equation 3.17 is true at all times, so we can once again integrate, this time with respect to the time $T$, giving

$$\int_0^t dT\ v_x(T) = \int_0^t dT\ (v_{x0} + AT). \tag{3.18}$$

or

$$x(t) - x(0) = v_0 t + \frac{1}{2}At^2 - (v_0 0 + \frac{1}{2}A0^2) \tag{3.19}$$

On the right-hand-side of this equation, I have shown explicitly the integrated terms at $T = 0$. Those terms are equal to zero in this problem, but in some problems the corresponding terms total to something that is non-zero. By convention the position $x(0)$ as time 0 is written $x_0$. Finally, in equation 3.17, $T$ is just a symbol for the time. We can replace the letter $T$ with any other letter that is not in use, without changing the meaning of the equation; in particular, we can replace $T$ with $t$. On rearranging the above equation, we finally have for motion in one dimension at constant acceleration

$$v_x(t) = v_{x0} + At, \tag{3.20}$$

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}At^2. \tag{3.21}$$

We have been very careful not to take any shortcuts in deriving these results from the constant acceleration starting point, equation 3.12, but the derivation is still far shorter than derivations that pretend to dodge calculus with various graphical methods. Furthermore, while you would have to fix a few steps, a slight variation on the above will give you $x(t)$ for an acceleration that depends on time. The key step is to recognize that if $f(t)$ is not a constant, then $\int dt\ f(t) \neq f(t)t$.

Some of you will have seen a peculiar equation $(v(t))^2 - (v(0))^2 = 2AX$ for motion at constant acceleration. Clearly that equation has to be consistent with the equations we just derived, but it's actually not a kinematic equation. This equation is a peculiar way to write the law of conservation of energy, for a particle under the influence of a constant force. *Most important!* In this equation $X$ is not the position of the particle at some time $t$. Instead, $X$ is the distance that the particle has travelled between times 0 and $t$. $X$ therefore has a completely different meaning than it does in equation 3.21. In terms of the $x(t)$ of equation 3.21, $X = x(t) - x(0)$. $X$ therefore *does not* tell you where the particle is; $X$ tells you (but only for this very special case) how far the particle has moved.

We now come to the hard part of the discussion. Question: What do the constants $x_0$ and $v_{0x}$ mean? You may have heard them called *initial* conditions, as though they were the position and velocity at the start of the problem, but the adjective "initial"' is to be charitable misleading. Let's consider a simple example that is actually not so simple.

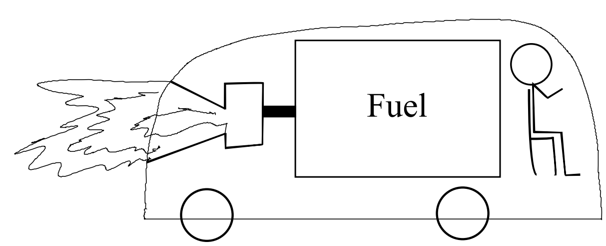


Figure 3.1: The Volksraketenwagon with engine at full thrust.

We advance to mythical California of 70 years ago and the quaint local custom of drag racing. Two cars driven by folks of limited good sense and even more limited regard for the law pull up at a red traffic light and stop. The character in the arrest-me-red sports car shouts at the character driving the apparent period SUV. *Hey, man, want to drag?*, this being an invitation to race the moment the light changes. When the light changes, the sports car takes off down the highway. The other vehicle seems to sit there. However, the other vehicle is not a period SUV. It is a highly modified vehicle, a *Volksraketenwagon* (People's Rocket Wagon), the modification being to replace everything behind the driver with a large liquid-fuel rocket engine hidden by the car's outer hull. The rocket takes ten seconds to power up, but then more or less instantly delivers a considerable constant thrust, to be precise, enough thrust that the Volksraketenwagon accelerates at $A = 100$ m/s$^2$. (That's ten gravities.) What are the motions of the Volksraketenwagon?

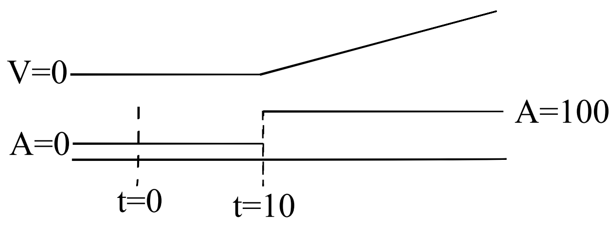


Figure 3.2: Velocity and acceleration of the Volksraketenwagon as functions of time.

A timeline illustrates what happens as time goes on. I've plotted the velocity and the acceleration as functions of time. The velocity curve for $t > 10$ is only qualitative. We are using SI units throughout, so I will not report the units every time. At time $t = 0$ and until $t = 10$ the vehicle is stationary. It is located at $x = 0$ and has $v_x(t) = 0$. At times $t \geq 10$ s, the vehicle accelerates at $A = 100$ m/s$^2$.

We are now going to advance by the Socratic method. I will ask questions. You get to decide on some answer, and *then* find out if you were right. To make this work, get out a piece of paper, and cover everything on the page below the solid line that follows this paragraph. Then lower the paper until you reach the next solid line. (You may have to flip to the next page to reach the next solid line.)

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Question: Can we describe the motion of the vehicle as seen on the entire the time line by inserting one set of numbers for $x_0$, $v_{0x}$, and $A$ into the equations

$$v_x(t) = v_{0x} + At, \tag{3.22}$$

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}At^2? \tag{3.23}$$

Stop and think. Your answer must be "Yes" or "No". When you are sure you have an answer, lower the paper again to the next line.

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The correct answer is "No". Why? Stop and think. When you have an answer, or give up, lower the paper to the next line.

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Why are these equations not usable as proposed? The equations describe motion at constant acceleration. However, along the length of the time line the acceleration is not constant. The acceleration has one value for $t < 10$ and a different value for $t > 10$. You need two sets of equations for motion at constant acceleration, one describing times $t < 10$ and the other describing times $t > 10$, and therefore two sets of constants $x_0$, $v_{0x}$, and $A_x$. If you realized the equations were OK if you used two different sets of constants for the two time ranges, so the problem is with the constants and not the equations, you understand well what the question meant.

Now let's work out what the constants are. I'll do this for you for $t < 10$. You should stop when you reach the next line, which is a distance down the page.

We know that at time $t = 0$ the volksraketenwagon is at $x(0) = 0$ and $v_x(0) = 0$, and has $A = 0$. Let's plug those numbers *carefully* into the above two equations. The two equations, with all zeros written explicitly, become

$$0 = v_{0x} + 0 \cdot 0, \tag{3.24}$$

$$0 = x_0 + v_{0x}0 + \frac{1}{2}0 \cdot 0^2? \tag{3.25}$$

In writing these equations, I replaced $t$ with its value zero. I carefully recalled that the velocity and position of the particle are $v_x(t)$ and $x(t)$, as found on the left hand sides of the two equations, and replaced them with their known values, 0 and 0 at time 0. I could equally well have used any other time $t < 10$. It would be wrong to replace $x_0$ and $v_{0x}$ with 0 and 0; you have to solve for $x_0$ and $v_{0x}$. However, if you have seen kinematics in earlier courses, you probably only saw problems in which doing the wrong thing, namely replacing $x_0$ and $v_{0x}$ with numbers in the problem, did not get you into trouble, so it appeared to be the right thing to do. The problems had been cooked so someone who had no clue what they were doing would still magically get the right answer. Back at the start of the book I had mentioned *praxis*, getting the right answer for the right reasons; here you are seeing an example of that term. The velocity and position are, however, $v_x(t)$ and $x(t)$, not $x_0$ and $v_{0x}$.

Solving first the upper equation for $v_{0x}$, we get $v_{0x} = 0$. Substituting that solution into the lower equation, we get $0 = x_0$. We have now solved for the constants of integration $v_{0x}$ and $x_0$. For $t < 10$, both constants are zero. We can now write for the velocity and position of the car at all times $t < 10$

$$v_x(t) = 0, \tag{3.26}$$

$$x(t) = 0. \tag{3.27}$$

Now you get to try the same process. Solve for $x_0$, $v_{x0}$, and $A$ for times $t > 10$. So that you all start at the same place, at $t = 10$ and after the rocket engine has fired, $A = 100$ but the car has barely moved, so it is still at location $x(10) \approx 0$ and stationary with $v_x(10) \approx 0$. OK, find equations for $x(t)$ and $v_x(t)$ for all times $t > 10$, meaning you need to find $x_0$ and $v_{0x}$ . Stop at the solid line following this pragraph, and do not go lower on the page until you have completed your work.

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Now move the paper down to the next line and read my answer

What did you find? Were your answers $x_0 = 0$ and $v_{0x} = 0$? After all, 0 and 0 were the initial values for the velocity and position. 0 and 0 are a fairly common pair of answers. Perhaps you found some other pair of numbers. What you should do, before saying you have the answer, is to check if your numbers are right. You do this by plugging them into the equations for $x(t)$ and $v_x(t)$ and solving for $v_x(10)$ and $x(10)$. If you did that, and thought $x_0 = 0$ and $v_{0x} = 0$, you should get

$$v_x(10) = 0 + 100 * 10, \tag{3.28}$$

$$x(10) = 0 + 0 * 10 + \frac{1}{2}100 * 10^2. \tag{3.29}$$

That is, if your answers for the constants of integration were 0 and 0, then, at the instant the rocket fired, the volksraketenwagon instantly gained a velocity of 1000 m/s and equally instantly transported itself to $x(10) = 5000$ m. That's not what happened. Something clearly went wrong. If you want, try again to find $x_0$ and $v_{0x}$.

How do you solve the problem correctly (some of you have already done this) for the three constants. (Three? remember $A$. I gave you $A = 100$.)

First, for the velocity at $t = 10$, the rocket having already ignited, one has

$$v(t) = v_{0x} + At, \tag{3.30}$$

$$0 = v_{0x} + 100 * 10, \tag{3.31}$$

$$v_{0x} = -1000. \tag{3.32}$$

This solution process has a few inobvious features. First, I always begin with the basic equation into which I will be substituting. That's the first line. I then replace algebraic symbols with numbers, without doing any arithmetic. There is a certain matter of taste involved in solving problems. I could equally well have solved first algebraically for $v_{0x}$, and then plugged in numbers, again without doing any arithmetic. That would have been equally correct. Then I counted equations and unknowns, to make sure that I have as many equations as there are unknowns. Here I have one unknown, and one equation. That's not a guarantee that the problem is soluble, but if you have fewer equations than you have unknowns, then you probably cannot solve for the unknown you want. Finally, I solve.

Now let's carry out the same process for $x_0$.

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}At^2, \tag{3.33}$$

$$x_0 = x(t) - v_{0x}t - \frac{1}{2}At^2, \tag{3.34}$$

$$x_0 = 0 - (-1000)10 - 0.5(100)10^2, \tag{3.35}$$

$$x_0 = 5000. \tag{3.36}$$

Why did I solve first for $v_{0x}$ and only then tried to solve for $x_0$? I looked at the two equations and realized that the equation for $v_x(t)$ contained only one unknown, so it could be solved immediately. The equation for $x(t)$ had two unknowns, one of which would come from the equation for $v_x(t)$, so I should solve for $x(t)$ second.

I again started with the basic equation to be solved. This time I first did a little algebra, so that $x_0$ was isolated algebraically before I plugged in numbers. Sometimes that step makes life simpler. Sometimes it doesn't. Experience will teach the difference. However, there is a useful anecdote here. Once upon a time, I had a colleague who assigned several students to do the same calculation, back in the days when algebraic calculations could only be done by hand. One of the students was notorious for doing things line by line, always taking only a very small step from one line to the next. The other students, who did several things on each line, were quite critical of him. He was slow! However, in the end, the patient student who always took small steps and checked his work after each step almost always got to the correct answer first. The other students sometimes eventually got to the right answer.

We now have $x_0$ and $v_{x0}$, so the equations for $v_x(t)$ and $x(t)$ should be

$$v_x(t) = -1000 + 100t, \tag{3.37}$$

$$x(t) = 5000 - 1000t + \frac{1}{2}100t^2. \tag{3.38}$$

You can check for yourself that these values for $x_0$ and $v_{0x}$ are correct, namely they show the volksraketen-wagon stationary at the origin at $t = 10$.

According to these equations, at time $t = 0$ the car was at $x = 5000$ m and was travelling at $v_x = -1000$ m/s, that is, the car was travelling at Mach three, three times the speed of sound. Backwards. What does this answer even mean?

Suppose the car always travelled at constant acceleration $A = 100$ m/s$^2$. In that case, at $t = 0$ it would indeed be 5 km out on front of the red light, headed backward toward the red light at Mach three. However, it would also have a positive acceleration. As a result, over the next 10 seconds its velocity would increase from -1000 m/s to 0 m/s. (Its speed would, by the same argument, be decreasing.) At time $t = 10$, the car would come to a stop, exactly at the red light, the location $x = 0$. It would then head away from the red light again, going in the positive $x$ direction, still accelerating at ten gravities. (Many years ago, when I described in lecture the car coming to a stop exactly at the red light, the description sounded very strange, but you can now watch on the internet something that is very much like the car. One of the commercial rocket launch corporations recovers its first stage rockets by having them return to ground and land vertically. If you find one of these videos, you can see the first stage descend, its rocket firing, and come to a stop just as it reaches the ground.)

We now have an example showing the meaning of the constants of integration $x_0$ and $v_{0x}$. $x_0$ is the position that the object would have had, at $t = 0$, if the object's acceleration had been constant from the times at which the information was given through to time zero. Yes, in the above case that's backward in time from $t = 10$ to $t = 0$. $v_{0x}$ is the velocity that the object would have had, at $t = 0$, if the object's acceleration had been constant from $t = 10$ to as far back in time as $t = 0$. Of course, the acceleration of the car was not constant from $t = 10$ back to time zero, so the $t > 10$ constants of integration in fact do not indicate the position and velocity of the car at $t = 0$.

It may occur to you to try shifting the time origin, so that $t = 10$ becomes $t' = 0$, and solving the problem. That's a special case approach. It fails, for example, if I give you the position of the rocket car at each of three unequally-spaced times. The approach above, sometimes with more algebraic work, is always effective.

Final point on the car problem: $x(t)$ is the position of the car. It is *not* how far the car has travelled. The distance the object has travelled between two times $t_1$ and $t_2$ is the object's *displacement*. The displacement between times $t_1$ and $t_2$ is $x(t_2) - x(t_1)$.

## 3.4 Discussion

Consider again what we just did. We have now seen a sketch of a general solution procedure:

Step one. Draw a picture of the situation. Insert in it all of the known quantities, and assign to each of them a symbol.

Step two. Write the basic equations. Identify which variables are unknown. Count that you have as many equations as you have unknowns. If you have fewer equations than you do unknowns, you will usually be unable to solve the equations for the unknowns.

Step three. Perhaps solve algebraically for the unknowns. Sometimes it is better to plug in numbers fairly early on in the calculation.

Step four. Plug in for the known variables and solve for the unknowns.

Aside: I am calling $dx/dt$ the *velocity*. Velocity is actually a vector, of which $dx/dt$ is a component. More on this later. Some authors call $dx/dt$ the *instantaneous velocity*. The word *instantaneous* is redundant, because the definition of a time derivative incorporates the limit $\delta t \to 0$. There is also an *average velocity*, which we will reach in the next chapter.

Finally, remember that my Solutions to the Worked Problems incorporate new material not seen elsewhere in the text.

## 3.5 Worked Problems

1. Standard notation: $\exp(a) \equiv e^a$. The position of an object as a function of time is given by $x(t) = x_0 t \exp(-at^{1/2})$. Here $x_0$ and $a$ are numerical constants in the equation for $x$. In this formula for $x(t)$, $x_0$ is not the position of the object at $t = 0$, and $a$ is not the acceleration of the object. Discuss the behavior of the velocity as $t \to 0+$ if $a > 0$.

2. A distinguished Faculty Member is loaded into a rocket ship.* The ship takes off from a pit at the bottom of Death Valley, starting at an altitude 100 m below Sea level. At $t = -5$ the engines are ignited. At $t = +5$s, the ship leaves the ground. The vertical acceleration of the ship during the climb is a constant. At $t = 20$ s the ship achieves an altitude of 4000 m above sea level. What is the ship's acceleration? What is its velocity at $t = 30$ s? Solve systematically, beginning with $z = z_0 + v_0 t + 0.5at^2$ and $v = v_0 + at$, not with whatever equation you may have pulled off the internet. Find $z_0$, $v_0$, and $a$. For credit, you must use my time and altitude origins in your calculations. To avoid undue complications, substitute numbers from the beginning. Prove your values for $z_0$, $v_0$ and $a$ are correct by showing that they predict correctly the locations and times supplied in the problem. Clue #1) I am fond of this problem, but I keep changing which boundary conditions I supply. A few years ago, 40% of the class got this one wrong. I hope you can do better. Clue #2 You should end up with three equations in three unknowns. You can do systematic elimination to find the unknowns, or you can learn how Mathematica, Maple, or some other computer algebra program will do your work for you. Clue #3) You should have performed the check "prove your values" automatically.

   *Yes, one of my former colleagues did go up in the space shuttle.

## 3.6 Homework

1. It is time for the first test of the atomic train, a steam locomotive in which the burning coal is replaced with a modest-sized block of radium. The engine is started. At some time, the locomotive begins to roll out of the station, maintaining a constant acceleration throughout its travels. After 2 miles, it goes off the rails, making a real mess and producing lawsuits beyond belief. You have been retained as the special physics consultant for a group of litigators. Your objective is to prove exactly where and when the train started.

   Fortunately, the railroad tracks are totally straight, and by agreement between the litigating parties they lie along the $x$ axis. Unfortunately the surviving cameras only report the locomotive's position at three moments of time, namely $x = 10$ at time $t = 10$, and later $x = -10$ at $t = 20$, and finally $x = -110$ at $t = 40$. (a) Report the position and velocity of the locomotive as explicit functions of time in the forms

$$x = x_o + v_o t + 0.5 a_x t^2, \tag{3.39}$$
$$v = v_o + a_x t. \tag{3.40}$$

(b) Confirm by appropriate substitutions that your answer agrees with the original information. (c) At the instant the train began to roll forward, where was it located? At what time did it begin to roll forward?

2. Our intrepid astronaut returns from a trip to outer space. At $t = -1$, the spaceship is at an altitude of 100 m above sea level. At $t = +1$, the spaceship is at an altitude of 20 m above sea level. At $t = +2$, the spaceship is at an altitude of 10 m above sea level. Write the altitude and vertical component of the velocity of the spaceship in the forms

$$z = z_0 + v_0 t + 0.5 a t^2, \tag{3.41}$$
$$v_z = v_0 + a t. \tag{3.42}$$

In a successful landing, the spaceship comes to a stop just as it touches the ground. Assuming that the landing was successful, what is the altitude of the ground? What was the altitude of the spaceship at $t = 0$?

3. Another Faculty Member is loaded into a rocket ship. The ship takes off from a mountain, starting at an altitude 1000 m above sea level. At $t = 0$ the engines are ignited. At $t = 7$s, the ship leaves the ground. The vertical acceleration of the ship during the climb is a constant. At $t = 30$ s the ship achieves an altitude of 5000 m above sea level. What is the ship's acceleration? What is its velocity at $t = 30$ s? Solve systematically, beginning with $z = z_0 + v_0 t + 0.5 a t^2$ and $v = v_0 + at$, not with whatever equation you may have pulled out from the book. Find $z_0$, $v_0$, and $a$. For credit, you must use my time and altitude origins in your calculations. To avoid undue complications, substitute numbers from the beginning. Prove your values for $z_0$, $v_0$ and $a$ are correct by showing that they predict correctly the locations and times supplied in the problem.

4. Yet another Faculty member is loaded into a rocket ship, on a secret launch pad in a secret location. At $t = -2$ the ship leaves the ground. The ship's vertical acceleration is constant after take-off. Radar shows that at $t = 2$ the rocket is at an altitude of -50 m relative to sea level. At $t = 10$ the ship has reached an altitude of 200 m. What was the ship's altitude at launch?

5. The rocket ship from the homework is being brought for a landing. It has a constant acceleration until it is brought to a stop, some distance above the ground. At times 1, 2, and 4, respectively, the rocket ship was at altitudes 91, 84, and 76 m, respectively. Write the altitude of the ship in the exact form

$$z(t) = z_0 + v_0 t + 0.5 a t^2, \tag{3.43}$$

including giving numerical values for $z_0$, $v_0$, and $a$.

6. Suppose the acceleration of a particle is given by $\frac{d^2x}{dt^2} = bt$, where $t$ is the time and $b$ is a constant. That is, the acceleration increases linearly with the time. Find $v_x(t)$ and $x(t)$.

7. Standard notation: $\exp(a) \equiv e^a$. The position of an object as a function of time is given by $x = x_0 t^3 \exp(-a t^{2/3})$. Here $x_0$ and $a$ are numerical constants in the equation for $x$. $x_0$ is not the position of the object at $t = 0$, and $a$ is not the acceleration of the object. Discuss the behavior of the velocity as $t \to 0+$ if $a > 0$.

8. Following equation 3.38, I said that you can check for yourself that you have the right answer here. Do the check. Show your work.

9. The position of an object as a function of time is given by $x = x_0 \exp(a t^{3/2})$. Find the velocity and the acceleration of the object. Discuss the behavior of the velocity and the acceleration as $t \to 0+$.

10. A local driver is proceeding along Salisbury Street in her BMW at an illegal speed of 50 m/s. To simplify the problem, Salisbury Street is taken to run along the $x$ axis, and the car is moving in the $+x$ direction. At $t = -8$, the vehicle's radar, laser, and sonar detectors fire, instantly triggering the brakes. The car slows with constant acceleration $a$. At $t = -4$, the car's speed is 10 m/s, still in the $+x$ direction. Between $t = -8$ and $t = -4$, the car's velocity in the $x$-direction may be written $v = v_0 + at$. (i) Show that the acceleration of the car is $a = -10$ m/s$^2$, and (ii) compute $v_0$.

11. A truck is approaching an intersection. The truck's motion is filmed and digitized. The supplied coordinates put the center of the intersection at $x = 15$. $x$ increases from left to right. At $t = -20$s, the truck appears on the film at $x = -75$ m with a speed of 15 m/s to the right. The brakes are applied at this time. The magnitude of the resulting acceleration is 2 m/s$^2$. (a) Give equations for $x(t)$ and $v(t)$ in the seconds after the brakes are applied, using equations in the form

$$x(t) = x_0 + v_0 t + 0.5at^2, \tag{3.44}$$

$$v(t) = v_0 + at. \tag{3.45}$$

(b) Prove your answers are correct by substituting $t = -20$ in these equations, and showing that you recover the initial position and velocity correctly. (c) How fast is the truck going when it reaches the intersection? (d) At what time does the truck stop?

Clue#1: Some years ago, almost a third of the class got this one wrong. That's much better than the year before, when over half the class was wrong.

Clue#2: Key question: what are the signs of the velocity and acceleration at $t = -20$? Some of you were taught that all numbers are positive, a wrong fact that I will attempt to unteach.

Clue#3: You should always check your answers. You should *automatically* have done the $t = -20$ substitution, without being told, to check your answer.

12. A distinguished faculty member is again loaded into a rocket ship. At $t = 0$, the engines are ignited. At $t = 5$s, the ship leaves its floating launch pad, located exactly at sea level. The vertical acceleration during the climb is a constant. At $t = 25$s the ship attains an altitude of 3000m. What is the ship's acceleration? What is its vertical velocity at $t = 25$ s? Write the ship's altitude as a function of time in the form $z = z_0 + v_0 t + 0.5 a_z t^2$.

13. A distinguished faculty member is once again loaded into a rocket ship. At $t = 0$, the engines are ignited. At $t = 4$s, the ship leaves its floating launch pad, located exactly at sea level. The vertical acceleration during the climb is a constant. At $t = 30$ s the ship attains an altitude of 5000 m. (a) What is the ship's acceleration? (b) Write the ship's altitude as a function of time in the form $z = z_0 + v_0 t + 0.5 a_z t^2$ (you will need to solve for $z_0$, $v_0$, and $a_z$). (c) Write the ship's velocity as a function of time as $v(t) = v_0 + a_z t$. (d) Substitute into your answers for (b) and (c) and confirm that your equations agree with the numbers provided in the problem, e.g., confirm $z(4) = 0$. You should have performed this step automatically. Find the ship's velocity at $t = 30$.

14. You have been hired as a consulting physics detective to find an object dropped from the world's first atomic train. The train was under computer control, giving exact control over the train's acceleration. At $t = 0$ the train was at rest. The train then had an acceleration $d^2x/dt^2 = 0.01$ m/s$^2$. At time $t = T$, the object was dropped; the engine was thrown into reverse, so that the acceleration of the train became $d^2x/dt^2 = -0.02$ m/s$^2$. The train came to a stop at a distance $x = L$ from the starting point. The time $T$ is unknown. The value of $L$ has been kept secret from you, because the missing object is the world's only neutrino bomb [1]. [Fortunately, the bomb did not detonate. Unfortunately, neutrino bombs are very small and hard to see, so a physical search of the complete track is impractical.] You are to calculate, in terms of $L$, the moment $T$ at which the bomb was dropped, and the location along the tracks at which the bomb was dropped. [1] As described in a 4/1/195x Los Alamos memo.

15. A massive pendulum swings back and forth at the end of a long chain. Its horizontal acceleration is measured to be $d^2x/dt^2 = a_0 \sin(\omega t)$. At $t = 0$, the pendulum is located at position $x = s$; at that moment in time it has $v = 0$. [Hint: Therefore, at $t = 0$ the pendulum must not be at the center of its swing.] Find the pendulum's velocity $v$ and its position $s$ as functions of time.

16. It is time for the trial run of the world's fourth atomic train. Because the last three atomic trains were destroyed in the course of past hour examinations, the crew has wisely been replaced with a robot controller. The train exits the yards with an acceleration 0.1 m/s$^2$ until it reaches a speed of 10 m/s. It continues at this speed in a straight line in the $+x$ direction until the command is sent to the robot to engage the brakes. At this point, instead of slowing down the train was observed to increase in speed at constant acceleration. When the train reached a speed of 40 m/s, it exited from its tracks and turned into scrap metal. Unfortunately, the instrument recorders on the train mostly failed, but the positions of the train *while it was accelerating* are known at three times. The times and positions are:

$$t = 10, x = 120, \tag{3.46}$$
$$t = 20, x = 230, \tag{3.47}$$
$$t = 40, x = 510. \tag{3.48}$$

i) Write the position and velocity of the train while it was accelerating in the forms

$$x = x_0 + v_0 t + 0.5at^2, \tag{3.49}$$
$$v_x = v_0 + at, \tag{3.50}$$

and supply the needed constants.

ii) At what time did the train begin its final acceleration?

iii) At what time did the train leave the tracks?

17. You are now the world's leading physics detective. Your client [Phillies, et fils, Insurers of MagLevs] brings you to the scene of an ingenious murder on a maglev train. A train carrying a single passenger and a number of crewmen rolled into the train station, going east to west and gradually slowing down. The train's engines provided a constant thrust for the entire period. The crew got off the train as it was rolling to a stop. Unfortunately for the passenger, the train's engine was left engaged, so the train rolled back out of the station, the passenger still on board, and continued to accelerate until the bomb in the passenger compartment detonated. The bomb exploded 2000 m east of the coordinate origin. Unfortunately, the automatic cameras that should have recorded everything were not working very well, so all you have is the location of the train at three times, namely: $x = 2500$ at $t = -50$, $x = 925$ at $t = 400$, and $x = 1125$ at $t = 500$.

    Your contract with the insurance company specifies that to get any points you must begin with $x = x_o + v_o t + \frac{1}{2}a_x t^2$ and $v = v_o + at$. Your insurer lets you work numerically rather than symbolically. You must determine (a) the position and velocity of the train as a function of time. (b) the instant at which the bomb exploded.

18. The intrepid Faculty colleague of a prior problem is given another chance to fly into space and return, this time on board a privately-launched spaceship. The launch altitude in central Asia is 2000 m above sea level. At $t = 5$, the engines are ignited. At $t = 10$, the ship leaves the ground. During the climb phase, acceleration is constant. At $t = 100$, the spaceship achieves an altitude of 30000 m above sea level. What is the ship's acceleration? What is the ship's velocity at $t = 100$? Solve systematically, beginning with $x = x_o + v_o t + 0.5at^2$, $v_x = v_o + a_x t$, or corresponding equations for the $y$ and $z$ directions. For credit, you must use my coordinate origins. Do not introduce, e.g., the average acceleration formula or the conservation of energy formula, unless you derive them first.

19. Our intrepid astronaut prepares for a second launch into space. This time, the shuttle has been given an improved engine. The launch occurs from a point 50m above mean sea level. At $t = 3$ seconds, the shuttle clears its launch pad and begins to climb. At $t = 5$, the shuttle has reached an altitude of 110m (above mean sea level). (i) Write the altitude and vertical component of the velocity of the shuttle in

the forms

$$x = x_0 + v_0 t + 0.5at^2 \tag{3.51}$$

$$v = v_0 + at. \tag{3.52}$$

(ii) Compute the altitude and vertical component of the shuttle's velocity at $t = 10$ seconds. (iii) Compute $v$ at time $t = 0$. What is the physical meaning of this number?

20. The rocket car discussed in one lecture is travelling in the $+\hat{i}$ direction at cruising speed, 1000 m/s. The car's flight recorder records position, speed, and acceleration at all times. As the car goes down the highway, sensors report that a traffic light dead ahead will turn red, and it will be necessary to stop to avoid running the light. The flight recorder later shows that the retro-rockets were fired at $t = 50$ s, when the car was at $x = 20,000$ m relative to the origin, and at $t = 70$s the speed of the car was down to 500 m/s. The retro-rockets give the car a constant acceleration, and stop firing when the car comes to a rest relative to the pavement. (i) At what time $t$ do the retro-rockets stop firing? (ii) Write the car's position as a function of time, in the form $x = x_o + v_o t + 0.5at^2$, using the coordinate system specified in the problem, for the period while the retro-rockets were firing. Where is the car when it comes to a stop? (iii) In the equation of part (ii) of this problem, what is the physical meaning of the constants $x_o$ and $v_o$?

## 3.7 Solutions to the Worked Problems

1. We start with $x(t) = x_o t \exp(-at^{-1/2})$ The velocity $v(t)$ is the first derivative of $x(t)$ with respect to time. Note that I added *with respect to time.* If I had just said "the derivative" I could have meant the derivative with respect to $x_o$ or $a$. Those are perfectly legitimate derivatives that are actually useful under some circumstances (but not this one). You need to specify with respect to what you are taking the derivative, and I did. Taking the derivative, we have

$$v(t) = x_o \exp(-at^{-1/2}) + x_o t \exp(-at^{-1/2}) \frac{d}{dt}(-at^{-1/2}), \tag{3.53}$$

where the product rule gave us the two terms and the chain rule led to the final derivative, which simplifies to

$$v(t) = x_o \exp(-at^{-1/2}) + \frac{x_o a}{2} t^{-1/2} \exp(-at^{-1/2}). \tag{3.54}$$

What is the behavior as $t \to 0$? The term $t^{-1/2}$ diverges. Its exponential, however, goes to zero. An exponential goes to zero more strongly than a polynomial in the same variable diverges, so

$$\lim_{t \to 0} v(t) = 0. \tag{3.55}$$

2. We should start by collecting all the information given in the problem. The rocket takes off at $t = 5$, at which point it is at altitude $x = -100$ and velocity $v = 0$. At time $t = 20$, it is at $x = 4000$. We are discussing motion at constant acceleration, for which we may write

$$x = x_o + v_o t + \frac{1}{2}at^2, \tag{3.56}$$

$$v = v_o + at. \tag{3.57}$$

We now take the known facts and insert them into these equations. I will do that without making any simplifications, so there is no doubt as to where the numbers in the equations came from. This is a good general first step, in that it lets you check exactly where you started.

$$-100 = x_o + 5v_o + \frac{1}{2}a5^2, \tag{3.58}$$

$$0 = v_o + 5a, \tag{3.59}$$

$$4000 = x_o + 20v_o + \frac{1}{2}a20^2. \tag{3.60}$$

The rocket started stationary ($v = 0$) at an altitude $x = -100$, but it would be entirely incorrect to write $x_o = -100$ or $v_o = 0$. $x_o$ and $v_o$ are constants of integration, not the "initial position" or the "initial velocity".

We now count equations and unknowns. There are three equations and three unknowns, so a solution may be possible.

If we subtract the first of these equations from the third, $x_o$ is eliminated from the calculation. It happens that subtraction is a simple step for these particular equations, but in fact subtraction (after multiplication if necessary by a constant) is a highly effective general process for reducing the apparent number of constants. The third equation becomes

$$4100 = 15v_o + 187.5a. \tag{3.61}$$

The first equation has been used to eliminate a constant, namely $x_o$; the first equation's problem-solving potentialities are thus exhausted. We now use the two remaining equations to eliminate $v_o$. Multiplying the second equation by 15 and subtracting from the third equation, we find

$$4100 - 0 = 15v_o - 15v_o + 187.5a - 75a, \tag{3.62}$$

which simplifies on dividing by $(187.5 - 75)$ to

$$a = 36.44\text{m/s}^2. \tag{3.63}$$

We have now found the acceleration. Substituting for $a$ in the second of the original equations, we get $v_o = -182.2$ m/s. $v_o$ is what the velocity would have been at time $t = 0$, before the rocket took off, a hypothetical event that did not happen. Finally, inserting these numbers in the first equation, one finds

$$x_o = -100 - 5(-182.5) - 12.5(36.44), \tag{3.64}$$

$$x_o = 35. \tag{3.65}$$

Therefore, we can write for the position and velocity

$$x = 357 - 182.2t + \frac{1}{2} \cdot 36.44t^2, \tag{3.66}$$

$$v = -182.2 + 36.44t. \tag{3.67}$$

Are we done? No! Always check your results! We still have to check that these equations actually give us the correct initial data. Indeed, if we substitute $t \to 5$, we find $x = -100$ and $v = 0$, while if we substitute $t \to 20$ we find $x = 4000$. Those are indeed the original conditions, so our solution is correct.

We could equally well have used subtraction to eliminate the constant $v_o$ or $a$. To eliminate $a$, we could multiply the first equation by $5/12.5$ and subtract the product from the second equation, and separately multiply the first equation by $20^2/5^2$ and subtract the product from the third equation. These two subtractions would give us two equations that depend on the unknowns $x_o$ and $v_o$, but would be independent of $a$.